

RELATIONS BETWEEN EXCEPTIONAL SETS FOR ADDITIVE PROBLEMS

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ABSTRACT. We describe a method for bounding the set of exceptional integers not represented by a given additive form in terms of the exceptional set corresponding to a subform. Illustrating our ideas with examples stemming from Waring’s problem for cubes, we show, in particular, that the number of positive integers not exceeding N , that fail to have a representation as the sum of six cubes of natural numbers, is $O(N^{3/7})$.

1. INTRODUCTION

Bounds on exceptional sets in additive problems can oftentimes be improved by replacing a conventional application of Bessel’s inequality with an argument based on the introduction of an exponential sum over the exceptional set, and a subsequent analysis of auxiliary mean values involving the latter generating function. Such a strategy underlies the earlier work concerning slim exceptional sets in Waring’s problem due to one or both of the present authors (see [18], [19], [20], [8], [21]). Exponential sums over sets defining the additive problem at hand are intrinsic to the application of the Hardy-Littlewood (circle) method that underpins such approaches. One therefore expects each application of such a method to be highly sensitive to the specific identity of the sets in question. Our goal in this paper is to present an approach which, for many problems, is relatively robust to adjustments in the identity of the underlying sets. We illustrate our conclusions with some consequences for Waring’s problem, paying attention in particular to sums of cubes.

In order to present our conclusions in the most general setting, we must introduce some notation. When $\mathcal{C} \subseteq \mathbb{N}$, we write $\overline{\mathcal{C}}$ for the complement $\mathbb{N} \setminus \mathcal{C}$ of \mathcal{C} within \mathbb{N} . When a and b are non-negative integers, it is convenient to denote by $(\mathcal{C})_a^b$ the set $\mathcal{C} \cap (a, b]$, and by $|\mathcal{C}|_a^b$ the cardinality of $\mathcal{C} \cap (a, b]$. Next, when $\mathcal{C}, \mathcal{D} \subseteq \mathbb{N}$, we define

$$\mathcal{C} \pm \mathcal{D} = \{c \pm d : c \in \mathcal{C} \text{ and } d \in \mathcal{D}\}.$$

As usual, we use $h\mathcal{D}$ to denote the h -fold sum $\mathcal{D} + \cdots + \mathcal{D}$. Also, we define $\Upsilon(\mathcal{C}, \mathcal{D}; N)$ to be the number of solutions of the equation

$$c_1 - d_1 = c_2 - d_2, \tag{1.1}$$

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with $c_1, c_2 \in (\mathcal{C})_{2N}^{3N}$ and $d_1, d_2 \in (\mathcal{D})_0^N$. The starting point for our analysis of exceptional sets is the inclusion

$$(\overline{\mathcal{A} + \mathcal{B}} - \mathcal{B}) \cap \mathbb{N} \subseteq \overline{\mathcal{A}}. \quad (1.2)$$

In §2 we both justify this trivial relation, and also apply it to establish a relation between the cardinalities of complements of sets that encapsulates the key ideas of this paper. The following theorem is a special case of Theorem 2.1 below, in which we obtain a conclusion with the sets in question restricted to collections of residue classes.

Theorem 1.1. *Suppose that $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$. Then for each natural number N , one has*

$$\left(|\mathcal{B}|_0^N |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \right)^2 \leq |\overline{\mathcal{A}}|_N^{3N} \Upsilon(\overline{\mathcal{A} + \mathcal{B}}, \mathcal{B}; N).$$

The conclusion of Theorem 1.1 is not particularly transparent, so it seems appropriate to outline its significance and implications. Note first that $|\overline{\mathcal{A}}|_N^{3N}$ counts the number of natural numbers in the interval $(N, 3N]$ that do not lie in \mathcal{A} , which is to say, the *exceptional set* corresponding to $(\mathcal{A})_N^{3N}$. Likewise, we see that $|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N}$ counts the number of natural numbers in the interval $(2N, 3N]$ that do not lie in $\mathcal{A} + \mathcal{B}$, and hence the exceptional set corresponding to $(\mathcal{A} + \mathcal{B})_{2N}^{3N}$. Observe next that in many situations of interest, it is possible to show that the number of solutions \mathbf{c}, \mathbf{d} of the equation (1.1), counted by $\Upsilon(\mathcal{C}, \mathcal{D}; N)$, is essentially dominated by the diagonal contribution with $c_1 = c_2$ and $d_1 = d_2$. Thus, under suitable circumstances, one finds that

$$\Upsilon(\overline{\mathcal{A} + \mathcal{B}}, \mathcal{B}; N) \ll |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} |\mathcal{B}|_0^N,$$

and then Theorem 1.1 delivers the bound

$$|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \ll |\overline{\mathcal{A}}|_N^{3N} / |\mathcal{B}|_0^N.$$

In this way, we are able to show that the exceptional set corresponding to $\mathcal{A} + \mathcal{B}$, in the interval $(2N, 3N]$, is smaller than that corresponding to \mathcal{A} , in $(N, 3N]$, by a factor $O(1/|\mathcal{B}|_0^N)$. With few exceptions, the scale of this improvement is well beyond the competence of more classical applications of the circle method.

The most immediate consequences of Theorem 1.1 concern additive problems involving squares or cubes. We begin with a cursory examination of the former problems in §2. It is convenient, when k is a natural number, to describe a subset \mathcal{Q} of \mathbb{N} as being a *high-density subset of the k th powers* when (i) one has $\mathcal{Q} \subseteq \{n^k : n \in \mathbb{N}\}$, and (ii) for each positive number ε , whenever N is a natural number sufficiently large in terms of ε , then $|\mathcal{Q}|_0^N > N^{1/k-\varepsilon}$. Also, when $\theta > 0$, we shall refer to a set $\mathcal{R} \subseteq \mathbb{N}$ as having *complementary density growth exponent smaller than θ* when there exists a positive number δ with the property that, for all sufficiently large natural numbers N , one has $|\overline{\mathcal{R}}|_0^N < N^{\theta-\delta}$.

Theorem 1.2. *Let \mathcal{S} be a high-density subset of the squares, and suppose that $\mathcal{A} \subseteq \mathbb{N}$ has complementary density growth exponent smaller than 1. Then,*

whenever $\varepsilon > 0$ and N is a natural number sufficiently large in terms of ε , one has

$$|\overline{\mathcal{A} + \mathcal{S}}|_{2N}^{3N} \ll N^{\varepsilon-1/2} |\overline{\mathcal{A}}|_N^{3N}.$$

In §2 we provide a slightly more general conclusion that captures, inter alia, the qualitative features of recent work on sums of four squares of primes (see [18], [6]). Following a consideration of some auxiliary mean values in §3, we advance in §4 to a discussion of additive problems involving cubes.

Theorem 1.3. *Let \mathcal{C} be a high-density subset of the cubes, and suppose that $\mathcal{A} \subseteq \mathbb{N}$ has complementary density growth exponent smaller than θ , for some positive number θ . Then, whenever $\varepsilon > 0$ and N is a natural number sufficiently large in terms of ε , one has the following estimates:*

(a) (exceptional set estimates for $\mathcal{A} + \mathcal{C}$)

$$\begin{aligned} |\overline{\mathcal{A} + \mathcal{C}}|_{2N}^{3N} &\ll N^{\varepsilon-1/6} |\overline{\mathcal{A}}|_N^{3N} + N^{\varepsilon-2} \left(|\overline{\mathcal{A}}|_N^{3N} \right)^3; \\ |\overline{\mathcal{A} + \mathcal{C}}|_{2N}^{3N} &\ll N^{\varepsilon-1/3} |\overline{\mathcal{A}}|_N^{3N} + N^{\varepsilon-1} \left(|\overline{\mathcal{A}}|_N^{3N} \right)^2; \end{aligned}$$

(b) (exceptional set estimates for $\mathcal{A} + 2\mathcal{C}$)

$$\begin{aligned} |\overline{\mathcal{A} + 2\mathcal{C}}|_{2N}^{3N} &\ll N^{\varepsilon-1/2} |\overline{\mathcal{A}}|_N^{3N} + N^{\varepsilon-4/3} \left(|\overline{\mathcal{A}}|_N^{3N} \right)^2, \text{ provided that } \theta \leq 1; \\ |\overline{\mathcal{A} + 2\mathcal{C}}|_{2N}^{3N} &\ll N^{\varepsilon-2/3} |\overline{\mathcal{A}}|_N^{3N}, \text{ provided that } \theta \leq \frac{13}{18}; \end{aligned}$$

(c) (exceptional set estimates for $\mathcal{A} + 3\mathcal{C}$)

$$\begin{aligned} |\overline{\mathcal{A} + 3\mathcal{C}}|_{4N}^{6N} &\ll N^{\varepsilon-5/3} \left(|\overline{\mathcal{A}}|_N^{6N} \right)^2, \text{ provided that } \theta \leq 1; \\ |\overline{\mathcal{A} + 3\mathcal{C}}|_{4N}^{6N} &\ll N^{\varepsilon-5/6} |\overline{\mathcal{A}}|_N^{6N}, \text{ provided that } \theta \leq \frac{8}{9}. \end{aligned}$$

The bounds supplied by Theorem 1.3(a) have direct consequences for the exceptional set in Waring's problem for sums of cubes. When s is a natural number and N is positive, write $E_s(N)$ for the number of positive integers not exceeding N that fail to be represented as the sum of s positive integral cubes. Thus, if we define $\mathcal{C} = \{n^3 : n \in \mathbb{N}\}$, then we have $E_s(N) = |\overline{s\mathcal{C}}|_0^N$. In §5 we establish the following estimates for $E_s(N)$.

Theorem 1.4. *Let τ be any positive number with $\tau^{-1} > 2982 + 56\sqrt{2833}$. Then one has*

$$E_4(N) \ll N^{37/42-\tau}, \quad E_5(N) \ll N^{5/7-\tau}, \quad E_6(N) \ll N^{3/7-2\tau}.$$

The estimate presented here for $E_4(N)$ is simply a restatement of Theorem 1.3 of Wooley [17], itself only a modest improvement on Theorem 1 of Brüdern [2]. Our bound for $E_5(N)$ may be confirmed by a classical approach employing Bessel's inequality, and indeed such a bound is reported in equation (1.3) of [3]. Our approach in this paper is simply to apply the first estimate of Theorem 1.3(a). Finally, the estimate for $E_6(N)$ provided by Theorem 1.4 is new, and may be compared with the bound $E_6(N) \ll N^{23/42}$ reported in equation (1.3)

of [3]. Note that $\frac{23}{42} > 0.5476$, whereas one may choose a permissible value of τ so that $\frac{3}{7} - 2\tau < 0.4283$. Of course, in view of Linnik's celebrated work [10], one has $E_s(N) \ll 1$ for $s \geq 7$.

An important strength of Theorem 1.3 is the extent to which it is robust to adjustments in the set \mathcal{C} of cubes to which it is applied. It is feasible, for example, to extract estimates for exceptional sets in the Waring-Goldbach problem for cubes. The complications associated with inherent congruence conditions are easily accommodated by simple modifications of our basic framework. In order to illustrate such ideas, when s is a natural number and N is positive, write $\mathcal{E}_6(N)$ for the number of even positive integers not exceeding N , and not congruent to $\pm 1 \pmod{9}$, which fail to possess a representation as the sum of 6 cubes of prime numbers. In addition, write $\mathcal{E}_7(N)$ for the number of odd positive integers not exceeding N , and not divisible by 9, which fail to possess a representation as the sum of 7 cubes of prime numbers, and denote by $\mathcal{E}_8(N)$ the number of even positive integers not exceeding N that fail to possess a representation as the sum of 8 cubes of prime numbers. A discussion of the necessity of the congruence conditions imposed here is provided in the preamble to Theorem 1.1 of [19]. By applying a variant of Theorem 1.3, in §5 we obtain the following upper bounds on $\mathcal{E}_s(N)$ ($6 \leq s \leq 8$).

Theorem 1.5. *One has*

$$\mathcal{E}_6(N) \ll N^{23/28}, \quad \mathcal{E}_7(N) \ll N^{23/42} \quad \text{and} \quad \mathcal{E}_8(N) \ll N^{3/14}.$$

For comparison, Theorem 1 of Kumchev [9] supplies the weaker bounds

$$\mathcal{E}_6(N) \ll N^{31/35}, \quad \mathcal{E}_7(N) \ll N^{17/28} \quad \text{and} \quad \mathcal{E}_8(N) \ll N^{23/84}.$$

We have more to say concerning the Waring-Goldbach problem, so we defer further consideration of allied conclusions to a future occasion.

As the final illustration of our methods, in §6 we consider Waring's problem for biquadrates. Since fourth powers are congruent to 0 or 1 modulo 16, a sum of s biquadrates must be congruent to r modulo 16, for some integer r satisfying $0 \leq r \leq s$. If n is the sum of $s < 16$ biquadrates and $16|n$, moreover, then $n/16$ is also the sum of s biquadrates. It therefore makes sense, in such circumstances, to consider the representation of integers n with $n \equiv r \pmod{16}$ for some integer r with $1 \leq r \leq s$. Define $Y_s(N)$ to be the number of integers n not exceeding N that satisfy the latter condition, yet cannot be written as the sum of s biquadrates.

Theorem 1.6. *Write $\delta = 0.00914$. Then one has*

$$Y_7(N) \ll N^{15/16-\delta}, \quad Y_8(N) \ll N^{7/8-\delta}, \quad Y_9(N) \ll N^{13/16-\delta}, \\ Y_{10}(N) \ll N^{3/4-2\delta}, \quad Y_{11}(N) \ll N^{5/8-2\delta}.$$

Here, the estimates for $Y_s(N)$ when $7 \leq s \leq 9$ follow from a classical application of Bessel's inequality, combined with the work of Vaughan [12] and Brüdern and Wooley [4] concerning sums of biquadrates. Such techniques would also yield the bounds $Y_{10}(N) \ll N^{3/4-\delta}$ and $Y_{11}(N) \ll N^{11/16-\delta}$, each

of which is inferior to the relevant conclusion of Theorem 1.6. We remark that superior estimates are available if one is prepared to omit the congruence class s modulo 16, or $s - 1$ and s modulo 16, from the integers under consideration for representation as the sum of s biquadrates. We refer the reader to Theorems 1.1 and 1.2 of the authors' earlier work [8] for details. Finally, we note that in view of Theorem 1.2 of Vaughan [12], one has $Y_s(N) \ll 1$ for $s \geq 12$.

In §7 we discuss further the abstract formulation of exceptional sets underlying Theorem 1.1, and consider the consequences of the most ambitious conjectures likely to hold for the additive theory of exceptional sets.

Throughout, the letter ε will denote a sufficiently small positive number. We use \ll and \gg to denote Vinogradov's well-known notation, implicit constants depending at most on ε , unless otherwise indicated. In an effort to simplify our analysis, we adopt the convention that whenever ε appears in a statement, then we are implicitly asserting that for each $\varepsilon > 0$, the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε .

2. THE BASIC INEQUALITY

Our goal in this section is to establish the upper bound presented in Theorem 1.1, illustrating this relation with the inexpensive conclusion recorded in Theorem 1.2. We begin by spelling out the inclusion (1.2). The proof is by contradiction. Let $n \in \overline{\mathcal{A} + \mathcal{B}}$ and $b \in \mathcal{B}$. Suppose, if possible, that $n - b \in \mathcal{A}$. Then there exists an element a of \mathcal{A} for which $n - b = a$, whence $n = a + b \in \mathcal{A} + \mathcal{B}$. But then $n \notin \overline{\mathcal{A} + \mathcal{B}}$, contradicting our initial hypothesis. We are therefore forced to conclude that $n - b \notin \mathcal{A}$, so that if $n - b \in \mathbb{N}$, then $n - b \in \overline{\mathcal{A}}$. In this way, we confirm that $(\overline{\mathcal{A} + \mathcal{B}} - \mathcal{B}) \cap \mathbb{N} \subseteq \overline{\mathcal{A}}$, as desired.

We establish Theorem 1.1 in a more general form useful in applications. In this context, when q is a natural number and $\mathfrak{a} \in \{0, 1, \dots, q - 1\}$, we define $\mathcal{P}_{\mathfrak{a}} = \mathcal{P}_{\mathfrak{a}, q}$ by

$$\mathcal{P}_{\mathfrak{a}, q} = \{\mathfrak{a} + mq : m \in \mathbb{Z}\}.$$

Also, we describe a set \mathcal{L} as being a *union of arithmetic progressions modulo q* when, for some subset \mathfrak{L} of $\{0, 1, \dots, q - 1\}$, one has

$$\mathcal{L} = \bigcup_{\mathfrak{l} \in \mathfrak{L}} \mathcal{P}_{\mathfrak{l}, q}.$$

In such circumstances, given a subset \mathcal{C} of \mathbb{N} and integers a and b , it is convenient to write

$$\langle \mathcal{C} \wedge \mathcal{L} \rangle_a^b = \min_{\mathfrak{l} \in \mathfrak{L}} |\mathcal{C} \cap \mathcal{P}_{\mathfrak{l}, q}|_a^b.$$

Theorem 2.1. *Suppose that $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$. In addition, let \mathcal{L} , \mathcal{M} and \mathcal{N} be unions of arithmetic progressions modulo q , for some natural number q , and suppose that $\mathcal{N} \subseteq \mathcal{L} + \mathcal{M}$. Then for each natural number N , one has*

$$\left(\langle \mathcal{B} \wedge \mathcal{L} \rangle_0^N |\overline{\mathcal{A} + \mathcal{B}} \cap \mathcal{N}|_{2N}^{3N} \right)^2 \leq q |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \Upsilon(\overline{\mathcal{A} + \mathcal{B}} \cap \mathcal{N}, \mathcal{B} \cap \mathcal{L}; N).$$

Proof. We begin by deriving a variant of the relation (1.2). Let N be a large natural number, and suppose that $\mathcal{L}, \mathcal{M}, \mathcal{N}$ satisfy the hypotheses of the statement of the theorem. We may suppose that there are sets $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \subseteq \{0, 1, \dots, q-1\}$ with the property that

$$\mathcal{M} = \bigcup_{\mathfrak{a} \in \mathfrak{A}} \mathcal{P}_{\mathfrak{a}}, \quad \mathcal{L} = \bigcup_{\mathfrak{b} \in \mathfrak{B}} \mathcal{P}_{\mathfrak{b}} \quad \text{and} \quad \mathcal{N} = \bigcup_{\mathfrak{c} \in \mathfrak{C}} \mathcal{P}_{\mathfrak{c}}.$$

Moreover, in view of the hypothesis $\mathcal{N} \subseteq \mathcal{L} + \mathcal{M}$, there exists a subset \mathfrak{D} of $\mathfrak{B} \times \mathfrak{A}$, with $\text{card}(\mathfrak{D}) \leq q$, satisfying the property that

$$\bigcup_{\mathfrak{c} \in \mathfrak{C}} \mathcal{P}_{\mathfrak{c}} = \bigcup_{(\mathfrak{b}, \mathfrak{a}) \in \mathfrak{D}} (\mathcal{P}_{\mathfrak{a}} + \mathcal{P}_{\mathfrak{b}}).$$

In particular, for each $\mathfrak{c} \in \mathfrak{C}$, there exists a pair $(\mathfrak{b}, \mathfrak{a}) \in \mathfrak{D}$ satisfying the property that $\mathcal{P}_{\mathfrak{c}} = \mathcal{P}_{\mathfrak{a}} + \mathcal{P}_{\mathfrak{b}}$.

Suppose now that $\mathfrak{c} \in \mathfrak{C}$, and that $(\mathfrak{b}, \mathfrak{a}) \in \mathfrak{D}$ satisfies the condition that $\mathcal{P}_{\mathfrak{c}} = \mathcal{P}_{\mathfrak{a}} + \mathcal{P}_{\mathfrak{b}}$. Let N be a large natural number, and suppose that $n \in (\overline{\mathcal{A} + \mathcal{B}} \cap \mathcal{P}_{\mathfrak{c}})_{2N}^{3N}$ and $b \in (\mathcal{B} \cap \mathcal{P}_{\mathfrak{b}})_0^N$. Then

$$n - b \in (N, 3N] \cap (\mathcal{P}_{\mathfrak{c}} - \mathcal{P}_{\mathfrak{b}}) = (N, 3N] \cap \mathcal{P}_{\mathfrak{a}},$$

and so the argument in the opening paragraph of this section shows that $n - b \in (\overline{\mathcal{A}} \cap \mathcal{P}_{\mathfrak{a}})_N^{3N}$. We therefore deduce that

$$(\overline{\mathcal{A} + \mathcal{B}} \cap \mathcal{P}_{\mathfrak{c}})_{2N}^{3N} - (\mathcal{B} \cap \mathcal{P}_{\mathfrak{b}})_0^N \subseteq (\overline{\mathcal{A}} \cap \mathcal{P}_{\mathfrak{a}})_N^{3N}, \quad (2.1)$$

whence

$$|\overline{\mathcal{A}} \cap \mathcal{P}_{\mathfrak{a}}|_N^{3N} \geq \text{card} \left((\overline{\mathcal{A} + \mathcal{B}} \cap \mathcal{P}_{\mathfrak{c}})_{2N}^{3N} - (\mathcal{B} \cap \mathcal{P}_{\mathfrak{b}})_0^N \right).$$

Next, write $\rho_{\mathfrak{b}\mathfrak{c}}(m)$ for the number of solutions of the equation $m = n - b$, with $n \in (\overline{\mathcal{A} + \mathcal{B}} \cap \mathcal{P}_{\mathfrak{c}})_{2N}^{3N}$ and $b \in (\mathcal{B} \cap \mathcal{P}_{\mathfrak{b}})_0^N$. An application of Cauchy's inequality shows that

$$\left(\sum_{\mathfrak{c} \in \mathfrak{C}} \sum_{1 \leq m \leq 3N} \rho_{\mathfrak{b}\mathfrak{c}}(m) \right)^2 \leq \left(\sum_{\mathfrak{c} \in \mathfrak{C}} \sum_{1 \leq m \leq 3N} 1 \right) \left(\sum_{\mathfrak{c} \in \mathfrak{C}} \sum_{1 \leq m \leq 3N} \rho_{\mathfrak{b}\mathfrak{c}}(m)^2 \right). \quad (2.2)$$

On recalling the definition of $\Upsilon(\mathcal{C}, \mathcal{D}; N)$ from the preamble to Theorem 1.1, we have

$$\begin{aligned} \sum_{\mathfrak{c} \in \mathfrak{C}} \sum_{1 \leq m \leq 3N} \rho_{\mathfrak{b}\mathfrak{c}}(m)^2 &= \sum_{\mathfrak{c} \in \mathfrak{C}} \Upsilon(\overline{\mathcal{A} + \mathcal{B}} \cap \mathcal{P}_{\mathfrak{c}}, \mathcal{B} \cap \mathcal{P}_{\mathfrak{b}}; N) \\ &\leq \Upsilon(\overline{\mathcal{A} + \mathcal{B}} \cap \mathcal{N}, \mathcal{B} \cap \mathcal{L}; N). \end{aligned}$$

Moreover, a moment's reflection confirms that

$$\begin{aligned} \sum_{\mathfrak{c} \in \mathfrak{C}} \sum_{1 \leq m \leq 3N} \rho_{\mathfrak{bc}}(m) &= \sum_{\mathfrak{c} \in \mathfrak{C}} \sum_{n \in (\overline{\mathcal{A}} + \overline{\mathcal{B}} \cap \mathcal{P}_{\mathfrak{c}})^{3N}_{2N}} \sum_{b \in (\mathcal{B} \cap \mathcal{P}_{\mathfrak{b}})^N_0} 1 \\ &\geq \min_{\mathfrak{b} \in \mathfrak{B}} |\mathcal{B} \cap \mathcal{P}_{\mathfrak{b}}|_0^N \sum_{n \in (\overline{\mathcal{A}} + \overline{\mathcal{B}} \cap \mathcal{N})^{3N}_{2N}} 1 \\ &= \langle \mathcal{B} \wedge \mathcal{L} \rangle_0^N |\overline{\mathcal{A} + \mathcal{B}} \cap \mathcal{N}|^{3N}_{2N}. \end{aligned}$$

Observe next that, in view of the relation (2.1), when $\rho_{\mathfrak{bc}}(m) \geq 1$, one has $m \in (\overline{\mathcal{A}} \cap \mathcal{P}_{\mathfrak{a}})^{3N}_N$. Thus one has the upper bound

$$\sum_{\substack{1 \leq m \leq 3N \\ \rho_{\mathfrak{bc}}(m) \geq 1}} 1 \leq \sum_{m \in (\overline{\mathcal{A}} \cap \mathcal{P}_{\mathfrak{a}})^{3N}_N} 1 \leq |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N},$$

whence

$$\sum_{\mathfrak{c} \in \mathfrak{C}} \sum_{\substack{1 \leq m \leq 3N \\ \rho_{\mathfrak{bc}}(m) \geq 1}} 1 \leq q |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N}.$$

The conclusion of the theorem follows on substituting these relations into (2.2). \square

The conclusion of Theorem 1.1 is immediate from the case $q = 1$ of Theorem 2.1, in which \mathcal{L} , \mathcal{M} and \mathcal{N} are each taken to be \mathbb{Z} . As a first illustration of the ease with which Theorems 1.1 and 2.1 may be applied to concrete problems, we now establish a theorem which implies Theorem 1.2 by using the strategy presented first in the proof of Theorem 1.1 of [18]. We first extend the notation introduced in the preamble to the statement of Theorem 1.2. Let \mathcal{L} be a union of arithmetic progressions modulo q , for some natural number q . When k is a natural number, we describe a subset \mathcal{Q} of \mathbb{N} as being a *high-density subset of the k th powers relative to \mathcal{L}* when (i) one has $\mathcal{Q} \subseteq \{n^k : n \in \mathbb{N}\}$, and (ii) for each positive number ε , whenever N is a natural number sufficiently large in terms of ε , then $\langle \mathcal{Q} \wedge \mathcal{L} \rangle_0^N \gg_q N^{1/k-\varepsilon}$. Also, when $\theta > 0$, we shall refer to a set $\mathcal{R} \subseteq \mathbb{N}$ as having *\mathcal{L} -complementary density growth exponent smaller than θ* when there exists a positive number δ with the property that, for all sufficiently large natural numbers N , one has $|\overline{\mathcal{R}} \cap \mathcal{L}|_0^N < N^{\theta-\delta}$.

Theorem 2.2. *Let \mathcal{L} , \mathcal{M} and \mathcal{N} be unions of arithmetic progressions modulo q , for some natural number q , and suppose that $\mathcal{N} \subseteq \mathcal{L} + \mathcal{M}$. Suppose also that \mathcal{S} is a high-density subset of the squares relative to \mathcal{L} , and that $\mathcal{A} \subseteq \mathbb{N}$ has \mathcal{M} -complementary density growth exponent smaller than 1. Then, whenever $\varepsilon > 0$ and N is a natural number sufficiently large in terms of ε , one has*

$$|\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}|^{3N}_{2N} \ll_q N^{\varepsilon-1/2} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N}.$$

Proof. Throughout the proof of this theorem, implicit constants may depend on q . Let N be a large natural number, and suppose that $\mathcal{L}, \mathcal{M}, \mathcal{N}$ satisfy the hypotheses of the statement of the theorem. Also, let \mathcal{S} be a high density subset of the squares relative to \mathcal{L} . Then, in particular, there is a subset \mathcal{T}

of \mathbb{N} for which $\mathcal{S} \cap \mathcal{L} = \{n^2 : n \in \mathcal{T}\}$. Consider also a subset \mathcal{A} of \mathbb{N} having \mathcal{M} -complementary density growth exponent smaller than 1. Write $P = [N^{1/2}]$. The quantity $\Upsilon(\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}, \mathcal{S} \cap \mathcal{L}; N)$ counts the number of solutions of the equation

$$n_1 - n_2 = x_1^2 - x_2^2, \quad (2.3)$$

with $n_1, n_2 \in (\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N})_{2N}^{3N}$ and $x_1, x_2 \in (\mathcal{T})_0^P$. There are plainly

$$|\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}|_{2N}^{3N} |\mathcal{T}|_0^P$$

solutions of this equation with $n_1 = n_2$ and $x_1^2 = x_2^2$. Given any one of the

$$O\left(\left(|\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}|_{2N}^{3N}\right)^2\right)$$

available choices of n_1 and n_2 with $n_1 \neq n_2$, meanwhile, one may apply an elementary estimate for the divisor function to show that there are $O(N^\varepsilon)$ possible choices for $x_1 - x_2$ and $x_1 + x_2$ satisfying (2.3), whence also for x_1 and x_2 . On noting that $|\mathcal{T}|_0^P = |\mathcal{S} \cap \mathcal{L}|_0^N$, we find that

$$\begin{aligned} \Upsilon(\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}, \mathcal{S} \cap \mathcal{L}; N) &\ll |\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}|_{2N}^{3N} |\mathcal{S} \cap \mathcal{L}|_0^N \\ &\quad + N^\varepsilon \left(|\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}|_{2N}^{3N}\right)^2. \end{aligned}$$

We substitute this last estimate into the conclusion of Theorem 2.1, and thereby deduce that

$$\begin{aligned} (\langle \mathcal{S} \wedge \mathcal{L} \rangle_0^N)^2 |\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}|_{2N}^{3N} &\ll |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} |\mathcal{S} \cap \mathcal{L}|_0^N \\ &\quad + N^\varepsilon |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} |\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}|_{2N}^{3N}. \end{aligned}$$

But since \mathcal{S} is a high-density subset of the squares relative to \mathcal{L} , and the set \mathcal{A} has \mathcal{M} -complementary density growth exponent smaller than 1, then there exists a positive number δ with the property that

$$N^\delta |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \ll N^{1-\delta} < (N^{1/2-\varepsilon})^2 \ll (\langle \mathcal{S} \wedge \mathcal{L} \rangle_0^N)^2.$$

In addition, one has

$$\frac{|\mathcal{S} \cap \mathcal{L}|_0^N}{(\langle \mathcal{S} \wedge \mathcal{L} \rangle_0^N)^2} \ll N^{\varepsilon-1/2}.$$

Thus we deduce that

$$|\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}|_{2N}^{3N} \ll N^{\varepsilon-1/2} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} + N^{\varepsilon-\delta} |\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}|_{2N}^{3N},$$

and the conclusion of the theorem follows at once. \square

The estimate claimed in Theorem 1.2 follows from Theorem 2.2 on putting $q = 1$ and taking \mathcal{L} , \mathcal{M} and \mathcal{N} each to be \mathbb{Z} .

3. AUXILIARY MEAN VALUES INVOLVING CUBES

Before applying Theorems 1.1 or 2.1 to additive problems involving cubes, it is necessary to establish some auxiliary mean value estimates in order to bound the expression $\Upsilon(\overline{\mathcal{A} + \mathcal{B}}, \mathcal{B}; N)$ relevant to our problems. This we accomplish in the present section.

Let \mathcal{L} and \mathcal{N} be unions of arithmetic progressions modulo q , for some natural number q . In addition, let \mathcal{C} be a high-density subset of the cubes relative to \mathcal{L} , and let \mathcal{A} be a subset of \mathbb{N} . Consider a large natural number N , and write $P = N^{1/3}$. Observe first that when $s \in \mathbb{N}$, the quantity $\Upsilon(\overline{\mathcal{A} + s\mathcal{C}} \cap \mathcal{N}, s(\mathcal{C} \cap \mathcal{L}); N)$ is bounded above by the number of solutions of the equation

$$n_1 - n_2 = \sum_{i=1}^s (x_i^3 - y_i^3), \quad (3.1)$$

with $n_1, n_2 \in (\overline{\mathcal{A} + s\mathcal{C}} \cap \mathcal{N})_{2N}^{3N}$ and $1 \leq i \leq s$. Write $\mathcal{Z}(N)$ for $(\overline{\mathcal{A} + s\mathcal{C}} \cap \mathcal{N})_{2N}^{3N}$ and Z for $\text{card}(\mathcal{Z}(N))$. Also, define the exponential sums

$$f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^3) \quad \text{and} \quad K(\alpha) = \sum_{n \in \mathcal{Z}(N)} e(n\alpha).$$

Here, as usual, we write $e(z)$ for $e^{2\pi iz}$. Then, on considering the underlying diophantine equation, it follows from (3.1) that

$$\Upsilon(\overline{\mathcal{A} + s\mathcal{C}} \cap \mathcal{N}, s(\mathcal{C} \cap \mathcal{L}); N) \leq \int_0^1 |f(\alpha)^{2s} K(\alpha)^2| d\alpha. \quad (3.2)$$

We begin by considering the situation in which $s = 1$.

Lemma 3.1. *One has*

$$\int_0^1 |f(\alpha)^2 K(\alpha)^2| d\alpha \ll P^\varepsilon (P^{3/2} Z + Z^{5/3})$$

and

$$\int_0^1 |f(\alpha)^2 K(\alpha)^2| d\alpha \ll PZ + P^{1/2+\varepsilon} Z^{3/2}.$$

Proof. We estimate the integral in question first by means of the Hardy-Littlewood method. When $a \in \mathbb{Z}$ and $r \in \mathbb{N}$, define the major arcs $\mathfrak{N}(r, a)$ by putting

$$\mathfrak{N}(r, a) = \{\alpha \in [0, 1) : |r\alpha - a| \leq P^{-2}\},$$

and then take \mathfrak{N} to be the union of the arcs $\mathfrak{N}(r, a)$ with $0 \leq a \leq r \leq P$ and $(a, r) = 1$. Also, write $\mathfrak{n} = [0, 1) \setminus \mathfrak{N}$. Next, define $\Upsilon(\alpha)$ for $\alpha \in [0, 1)$ by taking

$$\Upsilon(\alpha) = (r + P^3 |r\alpha - a|)^{-1},$$

when $\alpha \in \mathfrak{N}(r, a) \subseteq \mathfrak{N}$, and otherwise by putting $\Upsilon(\alpha) = 0$. Also, define the function $f^*(\alpha)$ for $\alpha \in [0, 1)$ by taking $f^*(\alpha) = P\Upsilon(\alpha)^{1/3}$. On referring to Theorems 4.1 and 4.2, together with Lemma 2.8, of Vaughan [13], one readily confirms that the estimate

$$f(\alpha) \ll f^*(\alpha) + P^{1/2+\varepsilon}$$

holds uniformly for $\alpha \in \mathfrak{N}$. An application of Weyl's inequality (see Lemma 2.4 of [13]), meanwhile, reveals that

$$\sup_{\alpha \in \mathfrak{N}} |f(\alpha)| \ll P^{3/4+\varepsilon}.$$

Thus we find that, uniformly for $\alpha \in [0, 1]$, one has

$$|f(\alpha)|^2 \ll f^*(\alpha)^2 + P^{3/2+\varepsilon}, \quad (3.3)$$

whence

$$\int_0^1 |f(\alpha)^2 K(\alpha)^2| d\alpha \ll P^{3/2+\varepsilon} I_1 + P^2 I_2, \quad (3.4)$$

where

$$I_1 = \int_0^1 |K(\alpha)|^2 d\alpha \quad \text{and} \quad I_2 = \int_0^1 \Upsilon(\alpha)^{2/3} |K(\alpha)|^2 d\alpha.$$

By Parseval's identity, one has $I_1 = Z$. Meanwhile, an application of Hölder's inequality combined with Lemma 2 of [1] shows that

$$\begin{aligned} I_2 &\ll \left(\int_0^1 \Upsilon(\alpha) |K(\alpha)|^2 d\alpha \right)^{2/3} \left(\int_0^1 |K(\alpha)|^2 d\alpha \right)^{1/3} \\ &\ll (P^{\varepsilon-3}(PZ + Z^2))^{2/3} Z^{1/3}. \end{aligned}$$

On substituting these estimates into (3.4), we deduce that

$$\int_0^1 |f(\alpha)^2 K(\alpha)^2| d\alpha \ll P^{3/2+\varepsilon} Z + P^\varepsilon (P^{2/3} Z + Z^{5/3}),$$

and the first conclusion of the lemma follows.

In order to confirm the second estimate, we begin by considering the underlying diophantine equation. One finds that the mean value in question is bounded above by $\sum_m Q(m)^2$, where $Q(m)$ denotes the number of solutions of the equation $x^3 + n = m$, with $1 \leq x \leq P$ and $n \in \mathcal{Z}(N)$. The desired conclusion therefore follows from a trivial modification of Theorem 1 of [5] (see, for example, Theorem 6.2 of [13] with $k = 3$, $j = 1$ and $\nu = 1$, or the case $k = 3$ and $j = 1$ of Lemma 6.1 below). \square

Next we consider the situation with $s = 2$.

Lemma 3.2. *One has*

$$\int_0^1 |f(\alpha)^4 K(\alpha)^2| d\alpha \ll P^2 Z + P^{11/6+\varepsilon} Z^2,$$

and

$$\int_0^1 |f(\alpha)^4 K(\alpha)^2| d\alpha \ll P^\varepsilon (P^{5/2} Z + P^2 Z^{3/2} + PZ^2).$$

Proof. On making use of a bound of Parsell based on the methods of Hooley (see Lemma 2.1 of [11], and also [7]), the argument of the proof of Lemma 10.3 of [19] supplies the first bound claimed in the lemma. We refer the reader to the discussion on pages 420 and 447 of [19] for amplification on this matter.

For the second bound, we again apply the Hardy-Littlewood method, and for this purpose it is convenient to employ the notation introduced in the course of the proof of Lemma 3.1. First, from (3.3), we deduce that

$$\int_0^1 |f(\alpha)^4 K(\alpha)^2| d\alpha \ll P^{3/2+\varepsilon} I_3 + P^2 I_4, \quad (3.5)$$

where

$$I_3 = \int_0^1 |f(\alpha)^2 K(\alpha)^2| d\alpha \quad \text{and} \quad I_4 = \int_0^1 \Upsilon(\alpha)^{2/3} |f(\alpha)^2 K(\alpha)^2| d\alpha.$$

From the second estimate of Lemma 3.1, one has $I_3 \ll PZ + P^{1/2+\varepsilon} Z^{3/2}$. Meanwhile, an application of Schwarz's inequality combined with Lemma 2 of [1] on this occasion shows that

$$\begin{aligned} I_4 &\ll \left(\int_0^1 \Upsilon(\alpha)^{4/3} |K(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_0^1 |f(\alpha)^4 K(\alpha)^2| d\alpha \right)^{1/2} \\ &\ll (P^{\varepsilon-3}(PZ + Z^2))^{1/2} \left(\int_0^1 |f(\alpha)^4 K(\alpha)^2| d\alpha \right)^{1/2}. \end{aligned}$$

On substituting these estimates into (3.5), we conclude that

$$\int_0^1 |f(\alpha)^4 K(\alpha)^2| d\alpha \ll P^{3/2+\varepsilon}(PZ + P^{1/2}Z^{3/2}) + P^4(P^{\varepsilon-3}(PZ + Z^2)),$$

and the second estimate of the lemma follows. \square

Although we are able to avoid explicit reference to the case $s = 3$ within this paper, it is useful for future reference to provide additional bounds of utility in this situation.

Lemma 3.3. *One has*

$$\int_0^1 |f(\alpha)^6 K(\alpha)^2| d\alpha \ll P^\varepsilon(P^4 Z + P^3 Z^2),$$

and

$$\int_0^1 |f(\alpha)^6 K(\alpha)^2| d\alpha \ll P^\varepsilon(P^{7/2} Z + P^{10/3} Z^2).$$

Proof. We begin by observing that the first estimate of the lemma is essentially the bound supplied by Lemma 6.2 of [19], and indeed that the argument employed to establish the latter suffices for our purposes. For the second bound, we once again apply the Hardy-Littlewood method, and employ the notation introduced in the course of the proof of Lemma 3.1. First, from (3.3), we deduce that

$$\int_0^1 |f(\alpha)^6 K(\alpha)^2| d\alpha \ll P^{3/2+\varepsilon} I_5 + P^2 I_6, \quad (3.6)$$

where

$$I_5 = \int_0^1 |f(\alpha)^4 K(\alpha)^2| d\alpha \quad \text{and} \quad I_6 = \int_0^1 \Upsilon(\alpha)^{2/3} |f(\alpha)^4 K(\alpha)^2| d\alpha.$$

From the first estimate of Lemma 3.2, one has $I_5 \ll P^2 Z + P^{11/6+\varepsilon} Z^2$. Meanwhile, an application of Hölder's inequality combined with a routine computation shows that

$$\begin{aligned} I_6 &\ll \left(K(0)^2 \int_0^1 \Upsilon(\alpha)^2 d\alpha \right)^{1/3} \left(\int_0^1 |f(\alpha)^6 K(\alpha)^2| d\alpha \right)^{2/3} \\ &\ll (P^{\varepsilon-3} Z^2)^{1/3} \left(\int_0^1 |f(\alpha)^6 K(\alpha)^2| d\alpha \right)^{2/3}. \end{aligned}$$

On substituting these estimates into (3.6), we conclude that

$$\int_0^1 |f(\alpha)^6 K(\alpha)^2| d\alpha \ll P^{3/2+\varepsilon} (P^2 Z + P^{11/6} Z^2) + P^6 (P^{\varepsilon-3} Z^2),$$

and the second bound of the lemma follows. \square

4. ADDITIVE PROBLEMS INVOLVING CUBES

Our goal in this section is the proof of Theorem 1.3, and this we achieve in Theorem 4.1 below. It is useful in applications to have available conclusions analogous to those of Theorem 1.3, though with additional congruence conditions present. We therefore spend a little extra effort to establish more general conclusions of this type.

Theorem 4.1. *Let \mathcal{L} , \mathcal{M} and \mathcal{N} be unions of arithmetic progressions modulo q , for some natural number q . Suppose also that \mathcal{C} is a high-density subset of the cubes relative to \mathcal{L} , and that $\mathcal{A} \subseteq \mathbb{N}$ has \mathcal{M} -complementary density growth exponent smaller than θ , for some positive number θ . Then, whenever $\varepsilon > 0$ and N is a natural number sufficiently large in terms of ε , one has the following estimates:*

(a) *when $\mathcal{N} \subseteq \mathcal{L} + \mathcal{M}$, then without any condition on θ , one has*

$$|\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \ll_q N^{\varepsilon-1/6} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} + N^{\varepsilon-2} \left(|\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \right)^3,$$

and

$$|\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \ll_q N^{\varepsilon-1/3} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} + N^{\varepsilon-1} \left(|\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \right)^2;$$

(b) *when $\mathcal{N} \subseteq 2\mathcal{L} + \mathcal{M}$, then provided that $\theta \leq 1$, one has*

$$|\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \ll_q N^{\varepsilon-1/2} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} + N^{\varepsilon-4/3} \left(|\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \right)^2,$$

and when $\theta \leq \frac{13}{18}$, one has

$$|\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \ll_q N^{\varepsilon-2/3} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N};$$

(c) *when $\mathcal{N} \subseteq 3\mathcal{L} + \mathcal{M}$, then provided that $\theta \leq 1$, one has*

$$|\overline{\mathcal{A} + 3\mathcal{C}} \cap \mathcal{N}|_{4N}^{6N} \ll_q N^{\varepsilon-5/3} \left(|\overline{\mathcal{A}} \cap \mathcal{M}|_N^{6N} \right)^2,$$

and when $\theta \leq \frac{8}{9}$, one has

$$|\overline{\mathcal{A} + 3\mathcal{C}} \cap \mathcal{N}|_{4N}^{6N} \ll_q N^{\varepsilon-5/6} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{6N}.$$

Proof. Let N be a large natural number, write $P = N^{1/3}$, and suppose that $\mathcal{A}, \mathcal{C}, \mathcal{L}, \mathcal{M}, \mathcal{N}$ satisfy the hypotheses of the statement of the theorem. In what follows, implicit constants may depend on q . We begin by considering the estimates claimed in part (a) of the theorem. Observe that since \mathcal{C} is a high-density subset of the cubes relative to \mathcal{L} , then $\langle \mathcal{C} \wedge \mathcal{L} \rangle_0^N \gg N^{1/3-\varepsilon}$, and hence it follows from Theorem 2.1 that

$$\left(N^{1/3-\varepsilon} |\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \right)^2 \ll |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \Upsilon(\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}, \mathcal{C} \cap \mathcal{L}; N).$$

Consequently, on making use of the relation (3.2) with $s = 1$ and the first estimate of Lemma 3.1, we deduce that

$$\begin{aligned} N^{2/3-\varepsilon} |\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} &\ll N^{1/2+\varepsilon} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \\ &+ N^\varepsilon |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \left(|\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \right)^{2/3}. \end{aligned}$$

From here it follows that

$$|\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \ll N^{\varepsilon-1/6} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} + \left(N^{\varepsilon-2/3} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \right)^3.$$

The first estimate of part (a) is thus confirmed.

If instead we apply (3.2) with $s = 1$ and the second estimate of Lemma 3.1, then we obtain the bound

$$\begin{aligned} N^{2/3-\varepsilon} |\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} &\ll N^{1/3} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \\ &+ N^{1/6+\varepsilon} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \left(|\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \right)^{1/2}, \end{aligned}$$

whence

$$|\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \ll N^{\varepsilon-1/3} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} + \left(N^{\varepsilon-1/2} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \right)^2.$$

This delivers the second estimate of part (a).

We next consider the estimates claimed in part (b). Observe first that by applying an elementary divisor function estimate, one confirms that the number of representations of a positive integer n , as the sum of two positive integral cubes, is $O(n^\varepsilon)$. It follows that when \mathcal{C} is a high-density subset of the cubes relative to \mathcal{L} , then $\langle 2\mathcal{C} \wedge 2\mathcal{L} \rangle_0^N \gg N^{2/3-\varepsilon}$. We therefore find from Theorem 2.1 that

$$\left(N^{2/3-\varepsilon} |\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \right)^2 \ll |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \Upsilon(\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}, 2(\mathcal{C} \cap \mathcal{L}); N).$$

Consequently, on making use of the relation (3.2) with $s = 2$ and the second estimate of Lemma 3.2, we deduce that

$$\begin{aligned} N^{4/3-\varepsilon} |\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} &\ll N^{5/6+\varepsilon} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \\ &\quad + N^{2/3+\varepsilon} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \left(|\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \right)^{1/2} \\ &\quad + N^{1/3+\varepsilon} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} |\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N}. \end{aligned}$$

We therefore deduce that when \mathcal{A} has \mathcal{M} -complementary density growth exponent smaller than 1, then there is a positive number δ with the property that

$$\begin{aligned} |\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} &\ll N^{\varepsilon-1/2} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} + N^{\varepsilon-\delta} |\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \\ &\quad + \left(N^{\varepsilon-2/3} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \right)^2. \end{aligned}$$

The first estimate of part (b) now follows.

If instead we apply (3.2) with $s = 2$ and the first estimate of Lemma 3.2, then we obtain the bound

$$\begin{aligned} N^{4/3-\varepsilon} |\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} &\ll N^{2/3} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \\ &\quad + N^{11/18+\varepsilon} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} |\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}|_{2N}^{3N}. \end{aligned}$$

Thus, when \mathcal{A} has \mathcal{M} -complementary density growth exponent smaller than $\frac{13}{18}$, then there is a positive number δ with the property that

$$|\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \ll N^{\varepsilon-2/3} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} + N^{\varepsilon-\delta} |\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N}.$$

The second estimate of part (b) is now immediate.

Finally, we turn our attention to the estimates claimed in part (c) of the theorem. We take $\mathcal{N}_0 = 2\mathcal{L} + \mathcal{M}$, so that \mathcal{N}_0 is a union of arithmetic progressions modulo q . Then the first estimate of part (b) implies that when \mathcal{A} has \mathcal{M} -complementary density growth exponent smaller than 1, then

$$|\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}_0|_{4N}^{6N} \ll N^{\varepsilon-1/2} |\overline{\mathcal{A}} \cap \mathcal{M}|_{2N}^{6N} + N^{\varepsilon-4/3} \left(|\overline{\mathcal{A}} \cap \mathcal{M}|_{2N}^{6N} \right)^2.$$

In particular, there is a positive number δ for which

$$|\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}_0|_{4N}^{6N} \ll N^{\varepsilon-1/2}(N^{1-\delta}) + N^{\varepsilon-4/3}(N^{1-\delta})^2 \ll N^{2/3-\delta}.$$

Thus, by summing over dyadic intervals, it follows that $\mathcal{A} + 2\mathcal{C}$ has \mathcal{N}_0 -complementary density growth exponent smaller than $\frac{2}{3}$. On noting that $\mathcal{N} \subseteq \mathcal{N}_0 + \mathcal{L} = 3\mathcal{L} + \mathcal{M}$, we deduce from the second estimate of part (a) that

$$\begin{aligned} |\overline{\mathcal{A} + 3\mathcal{C}} \cap \mathcal{N}|_{4N}^{6N} &\leqslant \left| (\overline{\mathcal{A} + 2\mathcal{C}} + \overline{\mathcal{C}}) \cap (\mathcal{N}_0 + \mathcal{L}) \right|_{4N}^{6N} \\ &\ll N^{\varepsilon-1/3} |\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}_0|_{2N}^{6N} + N^{\varepsilon-1} \left(|\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}_0|_{2N}^{6N} \right)^2 \\ &\ll N^{\varepsilon-1/3} |\overline{\mathcal{A} + 2\mathcal{C}} \cap \mathcal{N}_0|_{2N}^{6N}, \end{aligned}$$

and hence

$$|\overline{\mathcal{A} + 3\mathcal{C}} \cap \mathcal{N}|_{4N}^{6N} \ll N^{\varepsilon-5/6} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{6N} + \left(N^{\varepsilon-5/6} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{6N} \right)^2.$$

The first estimate of part (c) is now immediate.

Next we take $\mathcal{N}_1 = \mathcal{L} + \mathcal{M}$, so that \mathcal{N}_1 is a union of arithmetic progressions modulo q . The first estimate of part (a) implies that when \mathcal{A} has \mathcal{M} -complementary density growth exponent smaller than $\frac{8}{9}$, then

$$|\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}_1|_{4N}^{6N} \ll N^{\varepsilon-1/6} |\overline{\mathcal{A}} \cap \mathcal{M}|_{2N}^{6N} + N^{\varepsilon-2} \left(|\overline{\mathcal{A}} \cap \mathcal{M}|_{2N}^{6N} \right)^3.$$

In particular, there is a positive number δ for which

$$|\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}_1|_{4N}^{6N} \ll N^{\varepsilon-1/6} (N^{8/9-\delta}) + N^{\varepsilon-2} (N^{8/9-\delta})^3 \ll N^{13/18-\delta+\varepsilon}.$$

Thus, by summing over dyadic intervals, it follows that the set $\mathcal{A} + \mathcal{C}$ has \mathcal{N}_1 -complementary density growth exponent smaller than $\frac{13}{18}$. On noting that $\mathcal{N} \subseteq \mathcal{N}_1 + 2\mathcal{L} = 3\mathcal{L} + \mathcal{M}$, we deduce from the second estimate of part (b) that

$$\begin{aligned} |\overline{\mathcal{A} + 3\mathcal{C}} \cap \mathcal{N}|_{4N}^{6N} &\leqslant \left| (\overline{\mathcal{A} + \mathcal{C}} + 2\overline{\mathcal{C}}) \cap (\mathcal{N}_1 + 2\mathcal{L}) \right|_{4N}^{6N} \\ &\ll N^{\varepsilon-2/3} |\overline{\mathcal{A} + \mathcal{C}} \cap \mathcal{N}_1|_{2N}^{6N} \\ &\ll N^{\varepsilon-5/6} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{6N} + N^{\varepsilon-8/3} \left(|\overline{\mathcal{A}} \cap \mathcal{M}|_N^{6N} \right)^3, \end{aligned}$$

and hence, for some positive number δ , one has

$$|\overline{\mathcal{A} + 3\mathcal{C}} \cap \mathcal{N}|_{4N}^{6N} \ll N^{\varepsilon-5/6} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{6N} + N^{\varepsilon-\delta}.$$

The second estimate of part (c) now follows at once. □

On taking $q = 1$ and $\mathcal{L}, \mathcal{M}, \mathcal{N}$ each to be \mathbb{Z} , the various conclusions of Theorem 4.1 suffice to establish Theorem 1.3.

5. CONSEQUENCES FOR SUMS OF CUBES

The estimates contained in Theorems 1.4 and 1.5 are straightforward corollaries of Theorems 1.3 and 4.1, as we now demonstrate.

The proof of Theorem 1.4. The estimate for $E_4(N)$ recorded in the statement of Theorem 1.4 is, as mentioned earlier, simply a restatement of Theorem 1.3 of [17]. Write $\nu = 2982 + 56\sqrt{2833}$. We set $\mathcal{C} = \{n^3 : n \in \mathbb{N}\}$ and $\mathcal{A} = 4\mathcal{C}$, and note that this first estimate yields the bound $|\overline{\mathcal{A}}|_N^{3N} \leqslant E_4(3N) \ll N^{37/42-\tau}$, for any positive number τ with $\tau^{-1} > \nu$. An application of the first estimate of Theorem 1.3(a) yields the bound

$$\begin{aligned} |\overline{\mathcal{A} + \mathcal{C}}|_{2N}^{3N} &\ll N^{\varepsilon-1/6} E_4(3N) + N^{\varepsilon-2} (E_4(3N))^3 \\ &\ll N^{5/7-\tau+\varepsilon} + N^{9/14-3\tau+\varepsilon}. \end{aligned}$$

Write $\lceil \theta \rceil$ for the least integer not smaller than θ , and define the integers N_j for $j \geq 0$ by means of the iterative formula

$$N_0 = \lceil \frac{1}{2}N \rceil, \quad N_{j+1} = \lceil \frac{2}{3}N_j \rceil \quad (j \geq 0). \quad (5.1)$$

In addition, define J to be the least positive integer with the property that $N_J = 2$, and note that $J = O(\log N)$. Then, whenever τ_1 is a positive number with $\tau_1^{-1} > \nu$, one has

$$E_5(N) \leq 3 + \sum_{j=1}^J |\overline{\mathcal{A} + \mathcal{C}}|_{2N_j}^{3N_j} \ll N^{5/7 - \tau_1}.$$

Next we put $\mathcal{A} = 5\mathcal{C}$, and note that the estimate just provided implies that $|\overline{\mathcal{A}}|_N^{3N} \leq E_5(3N) \ll N^{5/7 - \tau}$, for any positive number τ with $\tau^{-1} > \nu$. We now apply the second estimate of Theorem 1.3(a), obtaining the bound

$$\begin{aligned} |\overline{\mathcal{A} + \mathcal{C}}|_{2N}^{3N} &\ll N^{\varepsilon-1/3} E_5(3N) + N^{\varepsilon-1} (E_5(3N))^2 \\ &\ll N^{8/21 - \tau + \varepsilon} + N^{3/7 - 2\tau + \varepsilon}. \end{aligned}$$

Thus, whenever τ_1 is a positive number with $\tau_1^{-1} > \nu$, one deduces that

$$E_6(N) \leq 3 + \sum_{j=1}^J |\overline{\mathcal{A} + \mathcal{C}}|_{2N_j}^{3N_j} \ll N^{3/7 - 2\tau_1}.$$

This completes the proof of Theorem 1.4. \square

The proof of Theorem 1.5. When p is a prime number exceeding 7, one has

$$p^3 \equiv 1 \pmod{2}, \quad p^3 \equiv \pm 1 \pmod{9} \quad \text{and} \quad p^3 \equiv \pm 1 \pmod{7}.$$

If we put $\mathcal{C} = \{p^3 : p \text{ prime and } p > 7\}$, then it follows that for $s \geq 5$, one has $s\mathcal{C} \subseteq \mathcal{N}_s$, where we write

$$\begin{aligned} \mathcal{N}_5 &= \{n \in \mathbb{N} : n \equiv 1 \pmod{2}, n \not\equiv 0, \pm 2 \pmod{9}, n \not\equiv 0 \pmod{7}\}, \\ \mathcal{N}_6 &= \{n \in \mathbb{N} : n \equiv 0 \pmod{2}, n \not\equiv \pm 1 \pmod{9}\}, \\ \mathcal{N}_7 &= \{n \in \mathbb{N} : n \equiv 1 \pmod{2}, n \not\equiv 0 \pmod{9}\}, \\ \mathcal{N}_s &= \{n \in \mathbb{N} : n \equiv s \pmod{2}\} \quad (s \geq 8). \end{aligned}$$

Observe that our definition of the exceptional set in the Waring-Goldbach problem for cubes, given in the preamble to Theorem 1.5, may be recovered by putting $\mathcal{E}_s(N) = |\overline{s\mathcal{C}} \cap \mathcal{N}_s|_0^N$, a definition that we now extend to all natural numbers s .

Write

$$\mathcal{L} = \{l \in \mathbb{N} : l \equiv 1 \pmod{2}, l \equiv \pm 1 \pmod{9}, l \equiv \pm 1 \pmod{7}\}.$$

Then \mathcal{L} and \mathcal{N}_s ($s \geq 5$) are unions of arithmetic progressions modulo 126 satisfying the condition that $\mathcal{N}_{s+1} = \mathcal{L} + \mathcal{N}_s$ and $\mathcal{N}_{s+2} = 2\mathcal{L} + \mathcal{N}_s$ ($s \geq 5$). Moreover, it follows from the Prime Number Theorem in arithmetic progressions that $\langle \mathcal{C} \wedge \mathcal{L} \rangle_0^N \gg N^{1/3} (\log N)^{-1}$, so that \mathcal{C} is a high-density subset of the cubes relative to \mathcal{L} . Observe next that the proof underlying the first estimate

of Theorem 1 of Kumchev [9] shows that $\mathcal{E}_5(N) \ll N^{79/84-\nu}$, for some positive number ν . But $\mathcal{E}_s(N) = |\overline{s\mathcal{C}} \cap \mathcal{N}_s|_0^N$, and so it follows from the first bound of Theorem 4.1(a) that

$$\begin{aligned} |\overline{5\mathcal{C} + \mathcal{C}} \cap \mathcal{N}_6|_{2N}^{3N} &\ll N^{\varepsilon-1/6} |\overline{5\mathcal{C}} \cap \mathcal{N}_5|_N^{3N} + N^{\varepsilon-2} \left(|\overline{5\mathcal{C}} \cap \mathcal{N}_5|_N^{3N} \right)^3 \\ &\ll N^{\varepsilon-1/6} \mathcal{E}_5(3N) + N^{\varepsilon-2} (\mathcal{E}_5(3N))^3 \\ &\ll N^{65/84-\nu+\varepsilon} + N^{23/28-3\nu+\varepsilon}. \end{aligned}$$

Consequently, on making use again of the notation introduced in (5.1), it follows that

$$\mathcal{E}_6(N) \leq 3 + \sum_{j=1}^J |\overline{5\mathcal{C} + \mathcal{C}} \cap \mathcal{N}_6|_{2N_j}^{3N_j} \ll N^{23/28}.$$

Likewise, from the first bound of Theorem 4.1(b), one finds that

$$\begin{aligned} |\overline{5\mathcal{C} + 2\mathcal{C}} \cap \mathcal{N}_7|_{2N}^{3N} &\ll N^{\varepsilon-1/2} |\overline{5\mathcal{C}} \cap \mathcal{N}_5|_N^{3N} + N^{\varepsilon-4/3} \left(|\overline{5\mathcal{C}} \cap \mathcal{N}_5|_N^{3N} \right)^2 \\ &\ll N^{\varepsilon-1/2} \mathcal{E}_5(3N) + N^{\varepsilon-4/3} (\mathcal{E}_5(3N))^2 \\ &\ll N^{37/84-\nu+\varepsilon} + N^{23/42-2\nu+\varepsilon}. \end{aligned}$$

Consequently, one has

$$\mathcal{E}_7(N) \leq 3 + \sum_{j=1}^J |\overline{5\mathcal{C} + 2\mathcal{C}} \cap \mathcal{N}_7|_{2N_j}^{3N_j} \ll N^{23/42}.$$

Finally, from the first bound of Theorem 4.1(c), one finds that

$$\begin{aligned} |\overline{5\mathcal{C} + 3\mathcal{C}} \cap \mathcal{N}_8|_{4N}^{6N} &\ll N^{\varepsilon-5/3} \left(|\overline{5\mathcal{C}} \cap \mathcal{N}_5|_N^{6N} \right)^2 \\ &\ll N^{\varepsilon-5/3} (\mathcal{E}_5(6N))^2 \ll N^{3/14-2\nu+\varepsilon}. \end{aligned}$$

Then, one has

$$\mathcal{E}_8(N) \leq 3 + \sum_{j=1}^J |\overline{5\mathcal{C} + 3\mathcal{C}} \cap \mathcal{N}_8|_{2N_j}^{3N_j} \ll N^{3/14}.$$

This completes the proof of Theorem 1.5. \square

6. SUMS OF BIQUADRATES

Our bounds for $Y_{10}(N)$ and $Y_{11}(N)$ depend on a generalisation of the second estimate of Lemma 3.1 to k th powers, with $k \geq 3$. This variant of Davenport's bound we record in Lemma 6.1 below. We first introduce some further notation. Let k be a natural number with $k \geq 3$, let \mathcal{K} be a high-density subset of the k th powers, and write

$$g(\alpha) = \sum_{x \in (\mathcal{K})_0^N} e(\alpha x).$$

Also, let \mathcal{Z} be a subset of \mathbb{N} , write $Z = |\mathcal{Z}|_0^N$, and define the exponential sum

$$K(\alpha) = \sum_{n \in (\mathcal{Z})_0^N} e(n\alpha).$$

Lemma 6.1. *Let k be a natural number with $k \geq 3$, and suppose that $1 \leq j \leq k - 2$. Let N be a large natural number, and put $P = N^{1/k}$. Then one has*

$$\int_0^1 |g(\alpha)^{2^j} K(\alpha)^2| d\alpha \ll P^{2^j-1} Z + P^{2^j-\frac{1}{2}j-1+\varepsilon} Z^{3/2}. \quad (6.1)$$

Proof. We begin with the trivial observation that, on considering the underlying diophantine equations, one has

$$\int_0^1 |g(\alpha)^{2^j} K(\alpha)^2| d\alpha \leq \int_0^1 |f(\alpha)^{2^j} K(\alpha)^2| d\alpha,$$

where

$$f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k).$$

Next, let Δ_j denote the j th iterate of the forward differencing operator, so that whenever ϕ is a function of a real variable z , one has

$$\Delta_1(\phi(z); h) = \phi(z + h) - \phi(z),$$

and when $J \geq 1$, then

$$\Delta_{J+1}(\phi(z); h_1, \dots, h_{J+1}) = \Delta_1(\Delta_J(\phi(z); h_1, \dots, h_J); h_{J+1}).$$

It follows via a modest computation that when $1 \leq J \leq k$, then

$$\Delta_J(z^k; \mathbf{h}) = h_1 \dots h_J p_J(z; \mathbf{h}),$$

where p_J is a homogeneous polynomial in z and \mathbf{h} of total degree $k - J$, in which the coefficient of z^{k-J} is $k!/(k - J)!$. By the Weyl differencing lemma (see, for example, Lemma 2.3 of [13]), one has

$$|f(\alpha)|^{2^j} \leq (2P)^{2^j-j-1} \sum_{|h_1| < P} \dots \sum_{|h_j| < P} T_j,$$

where

$$T_j = \sum_{x \in I_j} e(\alpha h_1 \dots h_j p_j(x; \mathbf{h})),$$

and $I_j = I_j(\mathbf{h})$ denotes an interval of integers, possibly empty, contained in $[1, P]$. On recalling the definition of $K(\alpha)$, therefore, it follows from orthogonality that the integral on the left hand side of (6.1) is bounded above by the number of integral solutions of the equation

$$h_1 \dots h_j p_j(z; \mathbf{h}) = n_1 - n_2, \quad (6.2)$$

with $|h_i| < P$ ($1 \leq i \leq j$), $1 \leq z \leq P$ and $n_l \in (\mathcal{Z})_0^N$ ($l = 1, 2$), and with each solution being counted with weight $(2P)^{2^j-j-1}$.

There are $O(P^j)$ possible choices for z and \mathbf{h} with $h_1 \dots h_j = 0$. Given any one such choice, the equation (6.2) implies that $n_1 = n_2$. Next, when m is a

natural number and $|h_i| < P$ ($1 \leq i \leq j$), write $\rho(m; \mathbf{h})$ for the number of integral solutions of the equation

$$h_1 \dots h_j p_j(z; \mathbf{h}) + n = m, \quad (6.3)$$

with $1 \leq z \leq P$ and $n \in (\mathcal{Z})_0^N$. Then on isolating the solutions with $h_1 \dots h_j = 0$ discussed earlier, we have shown at this point that

$$\int_0^1 |g(\alpha)^{2^j} K(\alpha)^2| d\alpha \ll P^{2^j-1} Z + P^{2^j-j-1} S_1, \quad (6.4)$$

where

$$S_1 = \sum_{m, \mathbf{h}} \rho(m; \mathbf{h}),$$

in which the summation is over m and \mathbf{h} with $m \in (\mathcal{Z})_0^N$ and $1 \leq |h_i| < P$ ($1 \leq i \leq j$). Furthermore, an application of Cauchy's inequality reveals that

$$\left(\sum_{m, \mathbf{h}} \rho(m; \mathbf{h}) \right)^2 \leq \left(\sum_{m, \mathbf{h}} 1 \right) S_2,$$

where

$$S_2 = \sum_{m, \mathbf{h}} \rho(m; \mathbf{h})^2.$$

We therefore see that

$$S_1 \ll (P^j Z)^{1/2} S_2^{1/2}. \quad (6.5)$$

Next observe that S_2 counts the number of integral solutions of the system of equations

$$h_1 \dots h_j p_j(z_1; \mathbf{h}) + n_1 = h_1 \dots h_j p_j(z_2; \mathbf{h}) + n_2 = n_3, \quad (6.6)$$

with $1 \leq z_1, z_2 \leq P$, $1 \leq |h_i| < P$ ($1 \leq i \leq j$) and $n_l \in (\mathcal{Z})_0^N$ ($l = 1, 2, 3$). When $1 \leq j \leq k - 2$, the expression

$$h_1 \dots h_j (p_j(z_1; \mathbf{h}) - p_j(z_2; \mathbf{h})) \quad (6.7)$$

is a non-constant polynomial in z_1 and z_2 . In particular, given a solution $\mathbf{z}, \mathbf{h}, \mathbf{n}$ counted by S_2 with $n_1 = n_2$, then for each fixed choice of z_1 and \mathbf{h} , there are $O(1)$ possible choices for z_2 . The number of solutions of this type is therefore bounded above by a fixed positive multiple of the number of integral solutions of the equation (6.3) with $1 \leq z \leq P$, $1 \leq |h_i| < P$ ($1 \leq i \leq j$) and $n, m \in (\mathcal{Z})_0^N$. We conclude, therefore, that the number of solutions $\mathbf{z}, \mathbf{h}, \mathbf{n}$ counted by S_2 with $n_1 = n_2$ is $O(S_1)$.

Now consider a solution $\mathbf{z}, \mathbf{h}, \mathbf{n}$ counted by S_2 with $n_1 \neq n_2$. Since the polynomial (6.7) is divisible by $z_1 - z_2$, one finds that h_1, \dots, h_j and $z_1 - z_2$ are all divisors of the non-zero integer $n_1 - n_2$. Given any one of the $O(Z^2)$ possible choices for n_1 and n_2 with $n_1 \neq n_2$, therefore, an elementary estimate for the divisor function confirms that the number of choices for \mathbf{h} and $z_1 - z_2$ counted by S_2 is at most $O(P^\varepsilon)$. Fixing any one such choice of \mathbf{h} and $d = z_1 - z_2$, and noting that $1 \leq j \leq k - 2$, one finds from (6.6) that z_1 is determined from the polynomial equation

$$h_1 \dots h_j (p_j(z_1; \mathbf{h}) - p_j(z_1 - d; \mathbf{h})) = n_2 - n_1,$$

to which there are $O(1)$ solutions. Given any one such solution, the value of $z_2 = z_1 - d$ is fixed, as is the value of n_3 from (6.6). Thus we conclude that there are $O(P^\varepsilon Z^2)$ solutions of this type.

At this point, we have shown that $S_2 \ll S_1 + P^\varepsilon Z^2$, whence from (6.5), we have

$$S_1 \ll (P^j Z)^{1/2} (S_1 + P^\varepsilon Z^2)^{1/2}.$$

Consequently, we derive the upper bound

$$S_1 \ll P^j Z + P^{\frac{1}{2}j+\varepsilon} Z^{3/2},$$

and the conclusion of the lemma follows from (6.4). \square

We note that the estimate supplied by Lemma 6.1 is related to that found in an intermediate step of the proof of Theorem 1 of [5].

We are now equipped to discuss additive problems involving biquadrates. Rather than constraining ourselves to the proof of Theorem 1.6, we again record a more general estimate.

Theorem 6.2. *Let \mathcal{L} , \mathcal{M} and \mathcal{N} be unions of arithmetic progressions modulo q , for some natural number q , and suppose that $\mathcal{N} \subseteq 2\mathcal{L} + \mathcal{M}$. Suppose also that \mathcal{B} is a high-density subset of the biquadrates relative to \mathcal{L} , and that $\mathcal{A} \subseteq \mathbb{N}$. Then, whenever $\varepsilon > 0$ and N is a natural number sufficiently large in terms of ε , one has*

$$|\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}|_{2N}^{3N} \ll_q N^{\varepsilon-1/4} |\overline{\mathcal{A} \cap \mathcal{M}}|_N^{3N} + N^{\varepsilon-1} \left(|\overline{\mathcal{A} \cap \mathcal{M}}|_N^{3N} \right)^2.$$

Proof. Let N be a large natural number, and suppose that $\mathcal{L}, \mathcal{M}, \mathcal{N}$ satisfy the hypotheses of the statement of the theorem. Also, let \mathcal{B} be a high density subset of the biquadrates relative to \mathcal{L} . Then, in particular, there is a subset \mathcal{T} of \mathbb{N} for which $\mathcal{B} \cap \mathcal{L} = \{n^4 : n \in \mathcal{T}\}$. Consider also a subset \mathcal{A} of \mathbb{N} . Write $P = [N^{1/4}]$. The quantity $\Upsilon(\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}, 2(\mathcal{B} \cap \mathcal{L}); N)$ is bounded above by the number of solutions of the equation

$$n_1 - n_2 = x_1^4 + x_2^4 - x_3^4 - x_4^4,$$

with $n_1, n_2 \in (\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N})_{2N}^{3N}$ and $x_i \in (\mathcal{T})_0^P$ ($1 \leq i \leq 4$). Putting $\mathcal{K} = \mathcal{B}$ and $\mathcal{Z} = \overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}$, then in the notation associated with the statement of Lemma 6.1, we obtain the bound

$$\Upsilon(\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}, 2(\mathcal{B} \cap \mathcal{L}); N) \leq \int_0^1 |g(\alpha)^4 K(\alpha)^2| d\alpha.$$

The estimate supplied by Lemma 6.1 therefore yields the relation

$$\begin{aligned} \Upsilon(\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}, 2(\mathcal{B} \cap \mathcal{L}); N) &\ll N^{3/4} |\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}|_{2N}^{3N} \\ &\quad + N^{1/2+\varepsilon} \left(|\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}|_{2N}^{3N} \right)^{3/2}. \end{aligned}$$

We substitute this estimate into the conclusion of Theorem 2.1, and thereby deduce that

$$\begin{aligned} & (\langle 2\mathcal{B} \wedge 2\mathcal{L} \rangle_0^N)^2 |\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}|_{2N}^{3N} \\ & \ll N^{3/4} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} + N^{1/2+\varepsilon} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \left(|\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}|_{2N}^{3N} \right)^{1/2}. \end{aligned}$$

The number of representations of a positive integer n as the sum of two integral squares is at most $O(n^\varepsilon)$ (this result is classical). It follows that the number of representations of a positive integer n as the sum of two biquadrates is also $O(n^\varepsilon)$. Since \mathcal{B} is a high-density subset of the biquadrates relative to \mathcal{L} , we deduce that $\langle 2\mathcal{B} \wedge 2\mathcal{L} \rangle_0^N \gg N^{1/2-\eta}$ for every positive number η . Thus we arrive at the upper bound

$$\begin{aligned} |\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}|_{2N}^{3N} & \ll N^{\varepsilon-1/4} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \\ & + N^{\varepsilon-1/2} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \left(|\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}|_{2N}^{3N} \right)^{1/2}, \end{aligned}$$

whence

$$|\overline{\mathcal{A} + 2\mathcal{B}} \cap \mathcal{N}|_{2N}^{3N} \ll N^{\varepsilon-1/4} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} + \left(N^{\varepsilon-1/2} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \right)^2,$$

and the conclusion of the theorem follows. \square

The proof of Theorem 1.6. When $x \in \mathbb{N}$, one has $x^4 \equiv 0$ or $1 \pmod{16}$. Put $\mathcal{B} = \{x^4 : x \in \mathbb{N}\}$ and $\mathcal{L} = \{n \in \mathbb{N} : n \equiv 0 \text{ or } 1 \pmod{16}\}$. Also, when s is a natural number, write

$$\mathcal{N}_s = \{n \in \mathbb{N} : n \equiv r \pmod{16} \text{ with } 1 \leq r \leq s\}.$$

Then \mathcal{L} and \mathcal{N}_s ($s \geq 1$) are unions of arithmetic progressions modulo 16 satisfying the condition that $\mathcal{N}_{s+2} = \mathcal{N}_s + 2\mathcal{L}$.

Observe next that, when $s \geq 7$ and $1 \leq r \leq s$, a classical application of Bessel's inequality leads from the argument underlying the proof of Theorem 1.2 of [12], via Theorem 2 of [4], to the estimate

$$|\overline{s\mathcal{B}} \cap \mathcal{N}_s|_N^{3N} \leq Y_s(3N) \ll N^{1-(s-6)/16-\delta_1}, \quad (6.8)$$

in which $\delta_1 > 0.00914$. Here, we have implicitly applied Weyl's inequality for superfluous variables in the familiar manner. Although we will not go into details within this paper, the argument required in order to treat the major arcs, in the implicit application of the circle method, follows along the lines of that described in §3 of [8]. The key ingredient is Lemma 5.4 of [14], which allows for the successful analysis of a sixth moment involving four smooth and two classical biquadratic Weyl sums. This completes our sketch of the proof of the estimates recorded in Theorem 1.6 for $Y_s(N)$ when $7 \leq s \leq 9$.

We now provide an estimate for $Y_s(N)$ when $s = 10$ and 11 . The set \mathcal{B} is trivially a high-density subset of the biquadrates relative to \mathcal{L} , and so it follows

from Theorem 6.2 that

$$\begin{aligned} |\overline{8\mathcal{B} + 2\mathcal{B}} \cap \mathcal{N}_{10}|_{2N}^{3N} &\ll N^{\varepsilon-1/4} |\overline{8\mathcal{B}} \cap \mathcal{N}_8|_N^{3N} + N^{\varepsilon-1} \left(|\overline{8\mathcal{B}} \cap \mathcal{N}_8|_N^{3N} \right)^2 \\ &\ll N^{\varepsilon-1/4} Y_8(3N) + N^{\varepsilon-1} (Y_8(3N))^2. \end{aligned}$$

Then from (6.8) we see that

$$|\overline{10\mathcal{B}} \cap \mathcal{N}_{10}|_{2N}^{3N} \ll N^{5/8-\delta_1+\varepsilon} + N^{3/4-2\delta_1+\varepsilon}.$$

Thus, again making use of the notation from (5.1), and with $\delta = 0.00914$, one deduces that

$$Y_{10}(N) \leq 3 + \sum_{j=1}^J |\overline{10\mathcal{B}} \cap \mathcal{N}_{10}|_{2N_j}^{3N_j} \ll N^{3/4-2\delta}.$$

Meanwhile, when $s = 11$, in like manner Theorem 6.2 delivers the bound

$$|\overline{9\mathcal{B} + 2\mathcal{B}} \cap \mathcal{N}_{11}|_{2N}^{3N} \ll N^{\varepsilon-1/4} Y_9(3N) + N^{\varepsilon-1} (Y_9(3N))^2.$$

Then from (6.8) we see that

$$|\overline{11\mathcal{B}} \cap \mathcal{N}_{11}|_{2N}^{3N} \ll N^{9/16-\delta_1+\varepsilon} + N^{5/8-2\delta_1+\varepsilon}.$$

Thus, again making use of the notation from (5.1), and with $\delta = 0.00914$, one deduces that

$$Y_{11}(N) \leq 3 + \sum_{j=1}^J |\overline{11\mathcal{B}} \cap \mathcal{N}_{11}|_{2N_j}^{3N_j} \ll N^{5/8-2\delta}.$$

This completes the proof of Theorem 1.6. \square

7. FURTHER REMARKS ON ABSTRACT EXCEPTIONAL SETS

Since the formulation of exceptional sets underlying our statement of Theorem 1.1 would appear to be novel to the literature, it seems worthwhile to explore some alternative approaches and associated consequences.

We begin by providing a formulation of Theorem 1.1 which, though equivalent, is sometimes more transparent in its application. In this context, when \mathcal{C} and \mathcal{D} are subsets of \mathbb{N} , it is convenient to define $\widehat{\Upsilon}(\mathcal{C}, \mathcal{D}; N)$ to be the number of solutions of the equation (1.1) with $c_1, c_2 \in (\mathcal{C})_{2N}^{3N}$, $d_1, d_2 \in (\mathcal{D})_0^N$ and $c_1 \neq c_2$. We then have

$$\Upsilon(\mathcal{C}, \mathcal{D}; N) = |\mathcal{C}|_{2N}^{3N} |\mathcal{D}|_0^N + \widehat{\Upsilon}(\mathcal{C}, \mathcal{D}; N).$$

Let N be a large natural number, and suppose that $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$. Then, when

$$\widehat{\Upsilon}(\overline{\mathcal{A} + \mathcal{B}}, \mathcal{B}; N) \leq |\mathcal{B}|_0^N |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N},$$

it follows from Theorem 1.1 that

$$\left(|\mathcal{B}|_0^N |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \right)^2 \leq 2 |\overline{\mathcal{A}}|_N^{3N} |\mathcal{B}|_0^N |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N}.$$

Meanwhile, when

$$\widehat{\Upsilon}(\overline{\mathcal{A} + \mathcal{B}}, \mathcal{B}; N) > |\mathcal{B}|_0^N |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N},$$

then instead Theorem 1.1 yields

$$\left(|\mathcal{B}|_0^N |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \right)^2 < 2 |\overline{\mathcal{A}}|_N^{3N} \widehat{\Upsilon}(\overline{\mathcal{A} + \mathcal{B}}, \mathcal{B}; N).$$

We summarise this interpretation in the following theorem.

Theorem 7.1. *Suppose that $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$. Then for each natural number N , one has either*

$$|\mathcal{B}|_0^N |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \leq 2 |\overline{\mathcal{A}}|_N^{3N},$$

or else

$$\left(|\mathcal{B}|_0^N |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \right)^2 \leq 2 |\overline{\mathcal{A}}|_N^{3N} \widehat{\Upsilon}(\overline{\mathcal{A} + \mathcal{B}}, \mathcal{B}; N).$$

An immediate application of Theorem 7.1 relates to exceptional set problems involving prime numbers. Let \mathcal{A} be a subset of \mathbb{N} , and suppose that \mathcal{P} is a non-empty subset of the prime numbers. For a fixed non-zero value of the integer $n_1 - n_2$, standard sieve methods show that there is a positive number C with the property that the number of solutions of the equation $p_1 - p_2 = n_1 - n_2$, in prime numbers $p_1, p_2 \in (\mathcal{P})_0^N$, is at most

$$C \frac{N}{(\log N)^2} \prod_{\substack{p \text{ prime} \\ p|(n_1 - n_2)}} \left(\frac{p-1}{p-2} \right) \ll \frac{N \log \log N}{(\log N)^2}.$$

Thus we find that

$$\widehat{\Upsilon}(\overline{\mathcal{A} + \mathcal{P}}, \mathcal{P}; N) \ll \left(|\overline{\mathcal{A} + \mathcal{P}}|_{2N}^{3N} \right)^2 N \frac{\log \log N}{(\log N)^2}.$$

The second alternative of Theorem 7.1 therefore implies that

$$|\overline{\mathcal{A} + \mathcal{P}}|_{2N}^{3N} \ll \left(\frac{|\overline{\mathcal{A}}|_N^{3N}}{N(\log \log N)^{-1}} \right)^{1/2} \left(\frac{N(\log N)^{-1}}{|\mathcal{P}|_0^N} \right) |\overline{\mathcal{A} + \mathcal{P}}|_{2N}^{3N}.$$

In situations wherein $|\overline{\mathcal{A}}|_N^{3N} = o(N/\log \log N)$ and \mathcal{P} has positive Dirichlet density, this yields the estimate

$$|\overline{\mathcal{A} + \mathcal{P}}|_{2N}^{3N} = o \left(|\overline{\mathcal{A} + \mathcal{P}}|_{2N}^{3N} \right).$$

Thus one finds that $|\overline{\mathcal{A} + \mathcal{P}}|_{2N}^{3N} = 0$ for large enough values of N , a conclusion that plainly holds in much wider generality than this illustrative example suggests. Meanwhile, the first alternative of Theorem 7.1 yields the bound

$$|\overline{\mathcal{A} + \mathcal{P}}|_{2N}^{3N} \leq 2 |\overline{\mathcal{A}}|_N^{3N} / |\mathcal{P}|_0^N \ll \frac{\log N}{N} |\overline{\mathcal{A}}|_N^{3N}.$$

In particular, in the scenario under consideration, one finds that

$$|\overline{\mathcal{A} + \mathcal{P}}|_{2N}^{3N} = o \left(\frac{\log N}{\log \log N} \right),$$

an exceptionally slim exceptional set estimate.

Consider next problems wherein the set \mathcal{B} is a non-empty set of natural numbers supported on the values of a polynomial sequence of degree exceeding one. A divisor function estimate shows that

$$\widehat{\Upsilon}(\overline{\mathcal{A} + \mathcal{B}}, \mathcal{B}; N) \ll N^\varepsilon \left(|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \right)^2.$$

In such a situation, the first alternative of Theorem 7.1 supplies the bound

$$|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \leq 2 |\overline{\mathcal{A}}|_N^{3N} / |\mathcal{B}|_0^N. \quad (7.1)$$

In the alternative case, one finds that

$$|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \ll N^\varepsilon \left(|\overline{\mathcal{A}}|_N^{3N} (|\mathcal{B}|_0^N)^{-2} \right)^{1/2} |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N}.$$

Thus, provided that

$$|\mathcal{B}|_0^N > N^\delta \left(|\overline{\mathcal{A}}|_N^{3N} \right)^{1/2}, \quad (7.2)$$

for some fixed positive number δ , and all large values of N , then we deduce that $|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \ll N^{-\delta/2} |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N}$. The latter implies that $|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} = 0$ for large enough values of N . In either case, therefore, provided that the condition (7.2) holds, then for large values of N one has the upper bound (7.1).

It is natural to speculate concerning the true magnitude of the improvement in the exceptional set estimates available from the addition of a set \mathcal{B} . An example at one end of the spectrum is given by taking a set $\mathcal{A} \subseteq \mathbb{N}$ with the property that $\overline{\mathcal{A}}$ is supported on even numbers only, and $\mathcal{B} = \{0, 1\}$. Then $\mathcal{A} + \mathcal{B} = \mathbb{N}$, so that no matter what the cardinality of $(\overline{\mathcal{A}})_N^{3N}$ may be, one has $|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} = 0$ with $|\mathcal{B}|_0^N = 2$. In addition, if $\mathcal{A} \subseteq \mathbb{N}$ has the property that $|\mathcal{A}|_N^{3N} = o(N)$ as N tends to infinity, then $|\overline{\mathcal{A}}|_N^{3N} \gg N$, and a priori there is no reason to suppose that $|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N}$ is $o(N)$.

The typical situation is probably reflected by a heuristic argument based on the application of the Hardy-Littlewood method. On estimating the contribution anticipated from the major arcs, one is led to the following speculation.

Conjecture 7.2. *Suppose that $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$. Then one has*

$$\widehat{\Upsilon}(\overline{\mathcal{A} + \mathcal{B}}, \mathcal{B}; N) \ll N^{\varepsilon-1} \left(|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} |\mathcal{B}|_0^N \right)^2. \quad (7.3)$$

If we substitute the conjectured estimate (7.3) into the second alternative of Theorem 7.1, then one obtains the estimate

$$|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \ll N^{\varepsilon-1} |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} |\overline{\mathcal{A}}|_N^{3N}.$$

When \mathcal{A} has complementary density growth exponent smaller than 1, therefore, it follows that for some positive number δ , one has

$$|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \ll N^{\varepsilon-\delta} |\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N},$$

which for sufficiently large values of N implies that $|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} = 0$. The first alternative of Theorem 7.1, meanwhile, yields the bound

$$|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \leq 2 |\overline{\mathcal{A}}|_N^{3N} / |\mathcal{B}|_0^N.$$

We may therefore conclude as follows.

Corollary 7.3. *Suppose that \mathcal{A} and \mathcal{B} are non-empty subsets of \mathbb{N} , and that \mathcal{A} has complementary density growth exponent smaller than 1. Assume the validity of Conjecture 7.2. Then for all large values of N , one has*

$$|\overline{\mathcal{A} + \mathcal{B}}|_{2N}^{3N} \leq 2 |\overline{\mathcal{A}}|_N^{3N} / |\mathcal{B}|_0^N.$$

The validity of the conditional estimate of Corollary 7.3 would have far reaching consequences. Let $\mathcal{X} = \{n^k : n \in \mathbb{N}\}$. As usual, we define $G(k)$ to be the least natural number s with the property that all sufficiently large integers are the sum of at most s k th powers of natural numbers. Equivalently, the number $G(k)$ is the least natural number s for which $|s\mathcal{X}|_0^N = O(1)$ for all large N . Also, let $G_1(k)$ denote the least natural number s_1 for which $s_1\mathcal{X}$ has complementary density growth exponent smaller than 1. Thus, when $s \geq G_1(k)$, almost all natural numbers are the sum of at most s k th powers of natural numbers.

By repeated application of the conditional Corollary 7.3, one deduces that for $G_1(k) < t < G_1(k) + k$, one has

$$|\overline{t\mathcal{X}}|_{2N}^{3N} \ll \left| \overline{(t-1)\mathcal{X}} \right|_N^{3N} N^{-1/k},$$

whence $t\mathcal{X}$ has complementary density growth exponent smaller than

$$1 - (t - G_1(k))/k.$$

In this way, one finds that

$$G(k) \leq G_1(k) + k.$$

The methods of Wooley [15], [16], in combination with a classical application of Bessel's inequality, yield the estimate

$$G_1(k) \leq \frac{1}{2}k(\log k + \log \log k + 2 + o(1)),$$

from which we deduce the conditional upper bound

$$G(k) \leq \frac{1}{2}k(\log k + \log \log k + 4 + o(1)).$$

Of course, it seems likely that theoretical advances sufficient to establish Conjecture 7.2 would already yield an estimate of the shape $G(k) = O(k)$.

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